PMM U.S.S.R., Vol.44,pp.751-755 0021-8928/81/6 0751 $7.50/0

UDC 539.3

DUAL VARIATIONAL PROBLEMS FOR BOUNDARY FUNCTIONALS OF THE LINEAR THEORY OF ELASTICITY*

V. Ia. TERESHCHENKO

Dual variational problem for use with the problem of minimization of the boundary functionals of three-dimensional theory of elasticity, is formulated using the method of orthogonal expansions at the boundary of the region constructed in /1/. Solutions of the initial and the dual problem obtained yield the estimates for the error of the approximate solutions of the boundary value problems of the theory of elasticity.

1. In the linear theory of elasticity the Lagrange functional $J(\mathbf{u})$ (defined on the field of admissible displacements $\mathbf{u}$) and the Castigliano functional $I(\sigma)$ (defined on the field of admissible stresses σ) are both dual functionals. This means that the relation

$$\inf_{\mathbf{u}} J(\mathbf{u}) = \sup_\sigma I(\sigma)$$

holds. The relation can be used to obtain two-sided estimates of the energy functional. Below we show that the same result can be obtained for the boundary functionals of the theory of elasticity, from the orthogonal expansions of the dual pair of the trace spaces $W_2^{*1/2}(S) \times W_2^{-1/2}(S)$ (S is the boundary of a bounded region $G \subset E_3$ occupied by the elastic medium) constructed in /1/

$$W_2^{*1/2}(S) = P \otimes P^\oplus, \quad W_2^{-1/2}(S) = P \otimes P^\perp \tag{1.1}$$

Here $W_2^{*1/2}(S) \subset W^{1/2}(S)$ is the space of traces of the displacement vectors $\varphi_0(x)$ and $x \in G$, which are solutions of the problem

$$\mathbf{A}\varphi_0 = 0 \text{ in } G \left(\int_G \varphi_0\, dG = 0\right), \quad \mathbf{t}^{(\nu)}(\varphi_0)|_S = \mathbf{t}^{(\nu)}(\mathbf{u}^*)|_S \tag{1.2}$$

P is the subspace of traces of the displacement vectors $\mathbf{u}'(x)$ satisfying the boundary condition on the fixed part of the surface S

$$P^* = \{\mathbf{u}'' \in W_2^{*1/2}(S) \mid [\mathbf{u}'', u']_{1/2,S} = 0,\ \forall u' \in P\},\ P^\perp = \{t^{(\nu)}(\mathbf{u}'') \in W_2^{-1/2}(S) \mid \langle \mathbf{t}^{(\nu)}(\mathbf{u}''), u'\rangle = 0,\ \forall \mathbf{u}' \in P\} \tag{1.3}$$

$P^\perp = \mathbf{T}P^\oplus$ (see /1/, lemmas 3—5).

In (1.2) and (1.3) $\mathbf{A}$ is a vector operator of the anisotropic theory of elasticity /2/ and $\mathbf{u}^*(x)$ is an arbitrary displacement vector satisfying, in the region G, the equation of the theory of elasticity $\mathbf{A}\mathbf{u}^* = \mathbf{K}$ ($\mathbf{K}(x)$, $x \in G$ is the volume force vector) and the boundary condition on the free part of the surface S. We shall write this vector in the form of a sum $\mathbf{u}^* = \mathbf{u}_0 + \varphi_0$ where $\mathbf{u}_0(x)$ is the energy solution of the fundamental boundary value problem of the theory of elasticity /3/, φ_0 is a solution of (1.2), $\mathbf{t}^{(\nu)}(\mathbf{u}^*)$ is the stress vector acting on the plane of the surface S and $\mathbf{u}''(x)$ is the displacement vector satisfying the condition $\mathbf{A}\mathbf{u}'' = 0$ in G and the boundary condition on the free part of the surface S. For such vectors $\mathbf{u}''$ and $\mathbf{v}''$ the Betti formula /1/ yields

$$\int_S \mathbf{v}''\cdot\mathbf{t}^{(\nu)}(\mathbf{u}'')\, ds = 2\int_G W(\mathbf{v}'', \mathbf{u}'')\, dG, \quad \int_S \mathbf{u}''\cdot\mathbf{t}^{(\nu)}(\mathbf{u}'')\, ds = 2\int_G W(\mathbf{u}'', \mathbf{u}'')\, dG \geqslant 0 \tag{1.4}$$

The right-hand side of the first equation of (1.4) is a symmetric bilinear form /2/, and the right-hand side of the second equation represents the corresponding quadratic form positive for the vectors $\mathbf{u}''$ satisfying the condition

$$\int_G \mathbf{u}''\, dG = 0$$

$[,]_{1/2,S}$ denotes a scalar product in the space $W_2^{*1/2}(S)$, $\langle,\rangle$ is the ratio of the duality on $W_2^{1/2}(S) \times W_2^{-1/2}(S)$, $W_2^{1/2}(S)$ is the Sobolev— Slobodetskii space and $W_2^{-1/2}(S)$ its dual, and $\mathbf{T}$ is the isometry of $W_2^{*1/2}(S)$ on $W_2^{-1/2}(S)$ given, according to the Riesz theorem, by the relation /1/

*Prikl.Matem.Mekhan.,44,No.6,1053-1059,1980

$$[\mathbf{v}'', \mathbf{u}'']_{1/2, S} = (\mathbf{v}'', T\mathbf{u}'')_{0, S} = \langle \mathbf{v}'', \mathbf{t}^{(\nu)}(\mathbf{u}'')\rangle, \quad \forall \mathbf{v}'', \mathbf{u}'' \in W_2^{*1/2}(S) \tag{1.5}$$

In addition we have /1/

$$\| \mathbf{T}\mathbf{u}'' \|_{-1/2, S} = \| \mathbf{t}^{(\nu)}(\mathbf{u}'') \|_{-1/2, S} \leqslant c_0 \| \mathbf{u}'' \|_{1/2, S}, \ c_0 > 0 \tag{1.6}$$

where $(.)_{0,S}$ is a scalar product in $L_2(S)$ and $\|\cdot\|_{1/2,S}$, $\|\cdot\|_{-1/2,S}$ are the norms in $W_2^{1/2}(S)$ and $W_2^{-1/2}(S)$, respectively.

It was shown in /1/ that, using the orthogonal expansion (1.1), we can construct an energy solution $\mathbf{u}_0$ of e.g. the first problem of the theory of elasticity

$$\mathbf{A}\mathbf{u}_0 = \mathbf{K} \text{ in } \mathbf{G}, \mathbf{u}_0 |_S = 0 \quad (P = \{0\})$$

by using the projection φ_0 of an arbitrary vector $\mathbf{u}^*$ into the subspace $P^{\oplus}$ followed by subtraction from the vector $\mathbf{u}^*$: $\mathbf{u}_0 = \mathbf{u}^* - \varphi_0$, so that $\mathbf{u}_0 |_S \in P$. In this manner the projection $\varphi_0 \in P^{\oplus}$ minimizes the functional $\mathbf{F}_0(\mathbf{u}'') = |\mathbf{u}^* - \mathbf{u}''|_{1/2, S}$ on $\mathbf{u}'' \in P^{\oplus}$, i.e.

$$\inf F_0(\mathbf{u}'') = \inf_{\mathbf{u}'' \in P^{\oplus}} |\mathbf{u}^* - \mathbf{u}''|_{1/2, S} = |\mathbf{u}^* - \varphi_0|_{1/2, S}$$

The existence follows from the second expansion of (1.1), and the theorem of orthogonal projection /2/ is used to show that the dual variational problem has a solution.

2. The variational problem for the functional $F_0(\mathbf{u}'')$ is equivalent to the variational problem for the functional $|\mathbf{u}^* - \mathbf{u}''|^2_{1/2,S} - |\mathbf{u}^*|^2_{1/2,S}$, since $\mathbf{u}^*$ is a known element. Otherwise it is equivalent, by virtue of the relation

$$|\mathbf{u}^* - \mathbf{u}''|^2_{1/2, S} - |\mathbf{u}^*|^2_{1/2, S} = |\mathbf{u}''|^2_{1/2, S} - 2[\mathbf{u}^*, \mathbf{u}'']_{1/2, S}$$

to the variational problem for the functional

$$F(\mathbf{u}'') = [\mathbf{u}'', \mathbf{u}'']_{1/2, S} - 2[\mathbf{u}^*, \mathbf{u}'']_{1/2, S}, \quad \mathbf{u}'' \in P^{\oplus}$$

which can be written, taking into account the relation (1.5), in the form

$$F(\mathbf{u}'') = (\mathbf{u}'', T\mathbf{u}'')_{0, S} - 2\langle u'', \mathbf{t}^{(\nu)}(u^*)\rangle$$

where $\mathbf{t}^{(\nu)}(\mathbf{u}^*) \in W_2^{-1/2}(S)$ is a known element. The element $\varphi_0 \in P^{\oplus}$ (in the case of the first fundamental problem of the theory of elasticity we have $P^{\oplus} = \overline{W}_2^{-1/2}(S)$ /1/) minimizing the functional $F(\mathbf{u}'')$, satisfies the condition

$$(\mathbf{T}\varphi_0, \mathbf{u}'')_{0, S} \equiv \langle \mathbf{T}\varphi_0, \mathbf{u}''\rangle = \langle \mathbf{t}^{(\nu)}(\mathbf{u}^*), \quad \mathbf{u}''\rangle, \ \forall\, \mathbf{u}'' \in P^{\oplus} \tag{2.1}$$

and from this follows $\mathbf{T}\varphi_0 = \mathbf{t}^{(\nu)}(\mathbf{u}^*) \in P^{\perp}$.

In what follows we shall need the following assertion which rephrases a known theorem (see /4/, p.137) for the case of a linear, symmetric and positive operator $L \in (V \to V^*)$ with an inverse $L^{-1} \in (V^* \to V)$ where V is a reflexive Banach space. The functional $f(v) = {}^1/_2\langle Lv, v\rangle$ represents the potential of the operator L, the functional

$$f^*(v^*) = f^*(0) + {}^1/_2\langle v^*, L^{-1}v^*\rangle, \quad f^*(0) = -f(L^{-1}0) = 0 \tag{2.2}$$

is the potential of the operator L^{-1}, and the following relations hold for any $v \in V$, $v^* \in V^*$:

$$f(v) + f^*(v^*) - \langle v^* \ v\rangle \geqslant 0, \quad f(v) + f^*(Lv) - \langle Lv, v\rangle = 0 \tag{2.3}$$

Note. Since the operator $\mathbf{T}$ is an isometry of $W_2^{*1/2}(S)$ on $W_2^{-1/2}(S)$ it follows that the operator $\mathbf{T}^{-1}$ is also an isometry of $W_2^{-1/2}(S)$ on $W_2^{*1/2}(S)$, therefore $\mathbf{T}^{-1}0 = 0$.

Theorem 1. Let $\varphi_0 W_2^1(G)$ be a generalized solution of the problem (1.2). Let this solution exist /3/ and let the relation

$$\varphi_0|_S \in P^{\oplus}, \mathbf{t}^{(\nu)}(\varphi_0) + \mathbf{t}^{(\nu)}(\mathbf{u}^*) = 2\mathbf{T}\varphi_0 \in P^{\perp},$$

where $P^{\oplus}$ and $P^{\perp}$ are given by (1.3), hold /1/. Then, for the functionals

$$F(\mathbf{u}'') = (\mathbf{u}'', \mathbf{T}\mathbf{u}'')_{0, S} - 2\langle \mathbf{u}'', \mathbf{t}^{(\nu)}(\mathbf{u}^*)\rangle, \quad \Phi(\mathbf{t}^{(\nu)}(\mathbf{u}'')) = -{}^1/_4(\mathbf{t}^{(\nu)}(\mathbf{u}'') + \mathbf{t}^{(\nu)}(\mathbf{u}^*), \quad \mathbf{T}^{-1}(\mathbf{t}^{(\nu)}(\mathbf{u}'') + \mathbf{t}^{(\nu)}(\mathbf{u}^*)))_{0, S}$$

we have the following dual relations:

$$F(\varphi_0) = \inf_{\mathbf{u}'' \in P^{\oplus}} F(\mathbf{u}'') = \sup_{\mathbf{t}^{(\nu)}(\mathbf{u}'') \in P^{\perp}} \Phi(\mathbf{t}^{(\nu)}(\mathbf{u}'')) = \Phi(\mathbf{t}^{(\nu)}(\varphi_0))$$

(since here the lower (upper) edge is reached, we can replace inf (syp) by min (max)). According to (2.2), a functional conjugate tc $f(\mathbf{u}'') = {}^1/_2(\mathbf{u}'', \mathbf{T}\mathbf{u}'')_{0, S}$, is given by

$$f^*(\mathbf{t}^{(\nu)}(\mathbf{u}'')) = {}^1/_2(\mathbf{t}^{(\nu)}(\mathbf{u}''), \ \mathbf{T}^{-1}(\mathbf{t}^{(\nu)}(\mathbf{u}'')))_{0, S}$$

Clearly, $f^*(0) = -f(T^{-1}0) = -f(0) = 0$ (see the note above).

Here and henceforth the duality ratio $\langle , \rangle$ becomes identical to the scalar product $(,)_{0,S}$ in the Hilbert space $L_2(S)$ of elements of $W_2^{1/2}(S) \times W_2^{-1/2}(S)$.

The expressions for the functionals $F(u'')$ and $\Phi(t^{(\nu)}(u''))$ yield

$$2F(u'') - 2\Phi(t^{(\nu)}(u'')) = 4f(u'') - 4\langle u'', t^{(\nu)}(u^*)\rangle + f^*(t^{(\nu)}(u'') + t^{(\nu)}(u^*))$$

The first equation of (2.3) yields the inequality

$$f(u'') + f^*(t^{(\nu)}(u'') + t^{(\nu)}(u^*)) \geqslant \langle u'', t^{(\nu)}(u'') + t^{(\nu)}(u^*)\rangle \forall u'' \in P^{\oplus}, \quad t^{(\nu)}(u'') + t^{(\nu)}(u^*) \in P^{\perp}$$

and, taking it into account,we obtain

$$2F(u'') - 2\Phi(t^{(\nu)}(u'')) \geqslant 3f(u'') - 4\langle u'', t^{(\nu)}(u^*)\rangle + \langle u'', t^{(\nu)}(u'') + t^{(\nu)}(u^*)\rangle = 3f(u'') - 3\langle u'', t^{(\nu)}(u^*)\rangle + \tag{2.4}$$
$$(u'', Tu'')_{0,S} = \frac{5}{2}(u'', Tu'')_{0,S} - 3\langle u'', t^{(\nu)}(u^*)\rangle$$

Since we assume the positiveness of the functional $F(u'')$, it follows that

$$\frac{6}{5}|\langle u'', t^{(\nu)}(u^*)\rangle| < 2|\langle u'', t^{(\nu)}(u^*)\rangle| \leqslant (u'', Tu'')_{0,S}$$

Then from (2.4) we obtain

$$F(u'') - \Phi(t^{(\nu)}(u'')) \geqslant 0, \quad \forall u'' \in P^{\oplus}, \quad t^{(\nu)}(u'') \in P^{\perp} \tag{2.5}$$

Further we have

$$F(\varphi_0) - \Phi(t^{(\nu)}(\varphi_0)) = 2f(\varphi_0) - 2\langle \varphi_0, t^{(\nu)}(u^*)\rangle + \frac{1}{2}f^*(t^{(\nu)}(\varphi_0) + t^{(\nu)}(u^*)) = 2f(\varphi_0) - 2\langle \varphi_0, t^{(\nu)}(u^*)\rangle + \tag{2.6}$$
$$\frac{1}{2}f^*(2T\varphi_0) = 2f(\varphi_0) + 2f^*(T\varphi_0) - 2\langle \varphi_0, t^{(\nu)}(u^*)\rangle$$

Since $t^{(\nu)}(\varphi_0)|_S = t^{(\nu)}(u^*)|_S$ (see (1.2)) and $\langle \varphi_0, t^{(\nu)}(\varphi_0)\rangle = \langle \varphi_0, T\varphi_0\rangle$, using (2.3) we obtain from (2.6)

$$F(\varphi_0) - \Phi(t^{(\nu)}(\varphi_0)) = 2f(\varphi_0) + 2f^*(T\varphi_0) - 2\langle T\varphi_0, \varphi_0\rangle = 0$$

which, together with the inequality (2.5), proves the theorem. We note that the duality relations were proved in the same manner in /4/.

3. Let $\varphi_n \in W_2^1(G)$ denote the approximate value of the projection φ_0, constructed using the Ritz method and such, that

$$\lim|u^* - \varphi_n|_{1/2,S} = |u^* - \varphi_0|_{1/2,S} = \inf_{u'' \in P^{\oplus}} F_0(u''), \quad n \to \infty$$

Now we can show that /2/ the approximate solution $u_n = u^* - \varphi_n$ of the fundamental boundary value problem of the theory of elasticity tends to its exact solution $u_0 = u^* - \varphi_0$ (here $u_0 - u_n = \varphi_n - \varphi_0$) in the metric of $W^{*1/2}(S)$

$$\lim|\varphi_0 - \varphi_n|_{1/2,S} = \lim|u_0 - u_n|_{1/2,S} = 0, \qquad n \to \infty$$

and, by virtue of (1.5) we also have

$$\lim\langle \varphi_n - \varphi_0, \quad t^{(\nu)}(\varphi_n - \varphi_0)\rangle = \lim\langle u_0 - u_n, \quad t^{(\nu)}, \quad (u_0 - u_n)\rangle = 0, \quad n \to \infty$$

From (1.6) it follows that

$$\lim_{n \to \infty}\|t^{(\nu)}(\varphi_n) - t^{(\nu)}(\varphi_0)\|_{-1/2,S} = \lim\|t^{(\nu)}(u_0 - t^{(\nu)}(u_n)\|_{-1/2,S} = 0,$$

The relation (1.4) also implies the convergence

$$\lim 2\int_G W(\varphi_n - \varphi_0)\,dG = \lim 2\int_G W(u_0 - u_n)\,dG = 0, \quad n \to \infty$$

and, since the quadratic form

$$2\int_G W(\varphi)\,dG$$

on the vectors $\varphi \in W_2^1(G)$ satisfying the condition

$$\int_G \varphi\,dG = 0$$

is positive definite /2,3/

$$2\int_G W(\varphi)\,dG \geqslant c\,\|\varphi\|_{1,G}^2, \quad c>0, \quad \|\cdot\|_{1,G}=\|\cdot\|_{W_2^1(G)} \tag{3.1}$$

we also have the convergence

$$\lim \|\varphi_n-\varphi_0\|_{1,G}=\lim\|\mathbf{u}_0-\mathbf{u}_n\|_{1,G}=0, \qquad n\to\infty$$

Next we consider the problem of estimating the error of approximate solutions of the dual variational problems formulated in Theorem 1 (see /2/).

Theorem 2. Let $\varphi_n = W_2^1(G)\ (\varphi_n\,|_S \in P^{\oplus},\ \ \mathbf{t}^{(\nu)}(\varphi_n)\in P^{\perp})$ be the arbitrary Ritz approximations for φ_0. Then the following error estimates hold:

$$\|\mathbf{u}_0-\mathbf{u}_n\|_{1,G}=\|\varphi_n-\varphi_0\|_{1,G}\leqslant\Delta(\varphi_n),\quad \|\mathbf{u}_0-\mathbf{u}_n\|_{1/2,S}=\|\varphi_n-\varphi_0\|_{1/2,S}\leqslant c_1\Delta(\varphi_n)$$

$$\|\mathbf{t}^{(\nu)}(\mathbf{u}_0)-\mathbf{t}^{(\nu)}(\mathbf{u}_n)\|_{-1/2,S}=\|\mathbf{t}^{(\nu)}(\varphi_n)-\mathbf{t}^{(\nu)}(\varphi_0)\|_{-1/2,S}\leqslant c_2\Delta(\varphi_n),\quad \left(\Delta(\varphi_n)=\left\{\frac{1}{c}\left[F(\varphi_n)-\Phi(\mathbf{t}^{(\nu)}(\varphi_n))\right]\right\}^{1/2}\right)$$

The projection φ_0 sought on which

$$\inf_{\mathbf{u}''\in P^{\oplus}}|\mathbf{u}^*-\mathbf{u}''|_{1/2,S}$$

is attained, is a generalized solution, belonging to $W_2^1(G)$ of the problem (1.2) in the sense /3/

$$2\int_G W(\varphi_0,\varphi)\,dG-\int_S\varphi\cdot\mathbf{t}^{(\nu)}(\mathbf{u}^*)\,ds=0,\quad \forall\varphi\in W_2^1(G) \tag{3.2}$$

which also minimizes the functional

$$F(\varphi)=2\int_G W(\varphi)\,dG-2\int_S\varphi\cdot\mathbf{t}^{(\nu)}(\mathbf{u}^*)\,ds \quad \text{on} \quad \varphi\in W_2^1(G),\quad \int_G\varphi\,dG=0$$

For the difference of the functionals we have

$$\frac{1}{2}F(\varphi_n)-\frac{1}{2}F(\varphi_0)=\int_G W(\varphi_n)\,dG-\int_G W(\varphi_0)\,dG-\int_S(\varphi_n-\varphi_0)\,\mathbf{t}^{(\nu)}(\mathbf{u}^*)\,ds$$

from which, assuming in (3.2) $\varphi=\varphi_n-\varphi_0$, we obtain

$$\frac{1}{2}F(\varphi_n)-\frac{1}{2}F(\varphi_0)=\int_G W(\varphi_n)\,dG-\int_G W(\varphi_0)\,dG-2\int_G W(\varphi_0,\varphi_n-\varphi_0)\,dG=\int_G W(\varphi_n)\,dG-\int_G W(\varphi_0)\,dG-$$

$$2\int_G W(\varphi_0,\varphi_n)\,dG+2\int_G W(\varphi_0)\,dG=\int_G W(\varphi_n-\varphi_0)\,dG$$

From this by virtue of (3.1) we obtain the inequality

$$F(\varphi_n)-F(\varphi_0)>c\,\|\varphi_n-\varphi_0\|_{1,G}^2$$

Now, using the inequality $F(\varphi_0)>\Phi(\mathbf{t}^{(\nu)}(\varphi_n))$ which follows from the duality ratio of Theorem 1, we obtain the estimate

$$\|\varphi_n-\varphi_0\|_{1,G}\leqslant\Delta(\varphi_n)$$

From the inequality of the theorem on traces on S of the functions belonging to $W_2^1(G)$ in the metric $W_2^{1/2}(S)$, we also obtain the estimate

$$\|\varphi_n-\varphi_0\|_{1/2,S}\leqslant c_1\Delta(\varphi_n),\quad c_1>0$$

and by virtue of (1.6) we have

$$\|\mathbf{t}^{(\nu)}(\varphi_n)-\mathbf{t}^{(\nu)}(\varphi_0)\|_{-1/2,S}\leqslant c_2\Delta(\varphi_n),\quad c_2>0$$

which proves the theorem.

Using the relation (see (2.1))

$$(\mathbf{T}\varphi_n,\psi_k)_{0,S}=\langle\mathbf{t}^{(\nu)}(\mathbf{u}^*),\psi_k\rangle,\quad\forall\psi_k\in P^{\oplus},\quad \left(\varphi_n=\sum_{i=1}^n a_i\psi_i\right)$$

which must be satisfied by the approximate solution φ_n of the problem on the minimum of the functional $F(\mathbf{u}'')$ and the relation $\mathbf{t}^{(\nu)}(\varphi_n)+\mathbf{t}^{(\nu)}(\mathbf{u}^*)=2\mathbf{T}\varphi_n$ (which is regarded as a duality ratio on $W_2^{1/2}(S)\times W_2^{-1/2}(S)$), the right-hand sides of the estimates obtained can be reduced to the form suitable for the computations

$$F(\varphi_n) - \Phi(\mathbf{t}^{(\nu)}(\varphi_n)) = (\mathbf{T}\varphi_n, \varphi_n)_{0,S} - 2\langle \mathbf{t}^{(\nu)}(\mathbf{u}^*), \varphi_n\rangle + {}^1/_4(\mathbf{t}^{(\nu)}(\varphi_n) + \mathbf{t}^{(\nu)}(\mathbf{u}^*), \mathbf{T}^{-1}(\mathbf{t}^{(\nu)}(\varphi_n) + \mathbf{t}^{(\nu)}(\mathbf{u}^*)))_{0,S} =$$

$$(\mathbf{T}\varphi_n, \varphi_n)_{0,S} - 2\langle \mathbf{t}^{(\nu)}(\mathbf{u}^*), \varphi_n\rangle + {}^1/_4(2\mathbf{T}\varphi_n, \mathbf{T}^{-1}(2\mathbf{T}\varphi_n))_{0,S} = (\mathbf{T}\varphi_n, \varphi_n)_{0,S} - 2\langle \mathbf{t}^{(\nu)}(\mathbf{u}^*), \varphi_n\rangle + \langle \mathbf{T}\varphi_n, \varphi_n\rangle_{0,S} =$$

$$2(\mathbf{T}\varphi_n, \varphi_n)_{0,S} - 2\langle \mathbf{t}^{(\nu)}(\mathbf{u}^*), \varphi_n\rangle = 2\int_S \varphi_n[\mathbf{t}^{(\nu)}(\varphi_n) - \mathbf{t}^{(\nu)}(\mathbf{u}^*)]\,ds$$

From this follows, in particular,

$$\lim_{n\to\infty} 2\int_S \varphi_n[\mathbf{t}^{(\nu)}(\varphi_n) - \mathbf{t}^{(\nu)}(\mathbf{u}^*)]\,ds = 2\int_S \varphi_0[\mathbf{t}^{(\nu)}(\varphi_0) - \mathbf{t}^{(\nu)}(\mathbf{u}^*)]\,ds = 0$$

since (see (1.2)) $\mathbf{t}^{\mu}(\varphi_0)|_S = \mathbf{t}^{(\nu)}(\mathbf{u}^*)|_S$.

We note that following the method of determination of the scalar product $[\mathbf{u}', \mathbf{v}'']_{1/2, S}$ of the elements $\mathbf{u}', \mathbf{v}'' \in W_2^{*1/2}(S)$ (see (1.5)), we can also find the scalar product of the elements $\mathbf{t}^{(\nu)}(\mathbf{u}'')$ and $\mathbf{t}^{(\nu)}(\mathbf{v}'')$ from $W_2^{-1/2}(S)$ in accordance with the Riesz theorem (see note)as follows:

$$[\mathbf{t}^{(\nu)}(\mathbf{u}''), \mathbf{t}^{(\nu)}(\mathbf{v}'')]_{-1/2, S} = (\mathbf{t}^{(\nu)}(\mathbf{u}''), \mathbf{T}^{-1}(\mathbf{t}^{(\nu)}(\mathbf{v}'')))_{0,S}$$

with the corresponding norm

$$|\mathbf{t}^{(\nu)}(\mathbf{u}'')|_{-1/2, S} = \{[\mathbf{t}^{(\nu)}(\mathbf{u}''), \mathbf{t}^{(\nu)}(\mathbf{u}'')]_{-1/2, S}\}^{1/2}$$

Then the problem of maximization $\max \Phi(\mathbf{t}^{(\nu)}(\mathbf{u}''))\,(\mathbf{t}^{(\nu)}(\mathbf{u}'') \in P^{\perp})$ becomes equivalent to the problem of minimization

$$\max \Phi_{\perp}(\mathbf{t}^{(\nu)}(\mathbf{u}'')) = \max\left[-\frac{1}{4}|\mathbf{t}^{(\nu)}(\mathbf{u}'') + \mathbf{t}^{(\nu)}(\mathbf{u}^*)|^2_{-1/2, S}\right] = \min {}^1/_4|\mathbf{t}^{(\nu)}(\mathbf{u}'') + \mathbf{t}^{(\nu)}(\mathbf{u}^*)|^2_{-1/2, S}$$

the solution of which is represented by the projection of the element $\mathbf{t}^{(\nu)}(\mathbf{u}^*) \in W_2^{-1/2}(S)$ on $P^{\perp}$

$$\min {}^1/_4|\mathbf{t}^{(\nu)}(\mathbf{u}'') + \mathbf{t}^{(\nu)}(\mathbf{u}^*)|^2_{-1/2, S} = {}^1/_4|\mathbf{t}^{(\nu)}(\varphi_0) + \mathbf{t}^{(\nu)}(\mathbf{u}^*)|^2_{-1/2, S}, \quad (\mathbf{t}^{(\nu)}(\mathbf{u}'') \in P^{\perp})$$

A different approach to the study of variational problems for convex functionals of the theory of elasticity using the concepts of duality, was discussed by the author in /5/.

REFERENCES

1. TERESHCHENKO V.Ia. Method of orthogonal expansions on the domain boundary in three-dimensional problems of the linear theory of elasticity. PMM Vol.43, No.4, 1979.
2. MIKHLIN S.G., Variational Methods in Mathematical Physics. English translation, Pergamon Press, Book No. 10146, 1964.
3. TERESHCHENKO V.Ia., Generalization of the Trefftz method for three-dimensional elasticity problems. English translation, Pergamon Press, J. Comput. Mat., mat. Phys. Vol.16,No.4,1976.
4. GAEVSKII Kh., GREGER K. and ZAKHARIAS K. Nonlinear Operator Equations and Operator Differential Equations. Moscow, "Mir", 1978.
5. TERESHCHENKO V. Ia., On the convex functionals analogous to the generalized Tefftz functionals in variational problems of the theory of elasticity, PMM Vol.4, No.1, 1980.

Translated by L.K.